CHAPMAN UNIVERSITY
SCHMID COLLEGE OF SCIENCE AND TECHNOLOGY

The Mathematical Basis of General Relativity

Natanael Alpay, Trevor Kling, Dr. Mihaela Vajiac

Chapman University

Outline

In 1915, Albert Einstein introduced his theory of gravitation known as general relativity. The development of Riemannian geometry enabled the precise formulation of Einstein's general theory of relativity. In this project, we investigate the mathematical underpinnings of the theory to better understand the meaning and implications of these formulae. The central results of general relativity are the Einstein field equations, which relate the geometry of spacetime to the distribution and motion of matter within it. While frequently introduced purely through tensor manipulation, these equations have a rich mathematical basis that we explore in this poster. In general relativity, equations are defined over spacetime, a union of three spatial coordinates with a single time coordinate. From a Riemannian geometry perspective, spacetime is formally defined as a pseudo-Riemannian manifold, which is a differentiable manifold equipped with a nondegenerate metric tensor. To better understand this concept, we introduce the notion of manifolds, metrics, and a variety of operations on manifolds related to tensors. Additionally, spacetime is referred to as "curved," and to explain the implications of this statement we define notions of curvature on manifolds, such as the scalar curvature and the Ricci Curvature. We conclude by examining solutions to the Einstein equations and their implications for cosmology.

Tensor

A **tensor** of type (p, q) is a multilinear map defined on a vector V and its dual V^* over a field F

$$T : (V^*)^p \times (V)^q \to F. \tag{1}$$

Denote v_i the components of a vector v in V and v^i components of a covector v^* in V^*. Define the *components of a tensor* by considering its action on a set of basis vectors,

$$T_{i_1,\ldots,i_p,j_1,\ldots,j_q} = T(\mathbf{e}_i^*, \ldots, \mathbf{e}_p^*, \mathbf{e}_1, \ldots, e_q).$$

Manifolds

A **smooth manifold** of dimension N is a locally compact, paracompact Hausdorff space M together with:
(1) An open covering $\{U_i\}_{i\in I}$ of M
(2) A collection of continuous, one-to-one maps

$$\{\Psi_i : U_i \to \mathbb{R}^N;\ i \in I\}$$

such that $\Psi_i(U_i)$ is open in $\mathbb{R}^N$ and these maps are smoothly compatible; when $U_i \cap U_j \neq \varnothing$, then $\Psi_j \circ \Psi_i^{-1}$ is smooth.
The **tangent space** to a manifold M at a point $p \in M$, denoted by $T_p(M)$ is the set of all vectors which can be represented as the tangent vector to a curve at p. Similarly, the **cotangent space** of M at p is the dual space of $T_p(M)$, $T_p^*(M)$.
Going forward, all manifolds considered will be **Riemannian Manifolds**, which are manifolds equipped with a **Riemannian metric** g, a symmetric $(0, 2)$-tensor defined as

$$g_p : T_p(M) \times T_p(M) \to \mathbb{R}$$

locally at p. The Riemannian metric also defines a metric on any tensor bundle on M. Going forward we will omit the subscript p in definitions, but bear in mind these properties are still locally defined. We can use this metric tensor to raise or lower indices:

$$g_{ij}v^i = v_j \qquad g^{ij}v_i = v^j,$$

where g_{ij} is the matrix associated with the Riemannian metric and g^{ij} is the inverse of the associated metric.

Equations of Motion, Classically

Many quantities from classical mechanics can be defined in terms of tensors. The **tensor of matter** is given by

$$M^{ij} = \rho u_i u_j - g^{ij}p,$$

where ρ is the proper density, u^i is the ith component of the velocity, and p is the pressure between particles. Similarly, the **electromagnetic stress-energy tensor** is given by

$$E_{ij} = F_i^k F_{kj} - \frac{1}{2}g_{ij}F_s^k F_k^s,$$

where F_i^j and F_{ij} denote components of the electromagnetic field. Combining these two, we find the **total tensor**

$$T_{ij} = M_{ij} - E_{ij}.$$

This formulation allows for the equations of motion for these particles to be simply written as

$$\partial_j T_i^j = 0 \quad \text{or} \quad \partial_j T^{ij} = 0.$$

The Riemann Tensor

The **Riemann tensor** is a tensor of type $(0, 4)$ defined on $(T_p(M))^4$ with values in $\mathbb{R}$ denoted by $R(x, y, u, v)$ such that

$$R(x, y, u, v) = \begin{bmatrix} s(x, u) & s(x, v) \\ s(y, u) & s(y, v) \end{bmatrix},$$

where the tensor s is

$$s(x_1, x_2) = (a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_2x_1 + a_{22}x_2^2).$$

This can be thought of as a *tensor of curvature* for a two dimensional surface as it characterizes the deviation of a surface from its tangent plane. Similarly, the Riemann Tensor is the curvature for the 4-dimensional spacetime.
When we introduce the coordinate vectors $\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l$ and represent the vector arguments as $x = e_i x_i$, we can write

$$R_{i\,j;\,p\,q} = R(e_i, e_j, e_p, e_q).$$

Spacetime

A **spacetime** is a real 4-dimensional manifold (x, y, z, t) equipped with a global tensor field g. At every point p, this manifold has a tangent plane T_p consisting of all tangent vectors to the manifold with base point p. Additionally, a spacetime is **Lorentzian**, meaning that there is a choice of basis for T_p such that g_p has the matrix $\mathrm{diag}(1, 1, 1, -1)$. Since the spacetime is a 4-dimensional manifold, it must be embedded in at least a 10-dimensional base space.

Curved Space

In general relativity, we wish for our coordinates to dwell on the spacetime manifold. However, this manifold itself has curvature, and to make a consistent choice of basis vectors, we define $P(X^1, X^2, \ldots, X^N)$ as the point on the manifold, with $X^h(x^1, x^2, x^3, x^4)$ for coordinates X^i, and

$$\mathbf{c}_i = \partial_i P = (\partial_i X^h)_h \quad (h = 1, \ldots, N; i = 1, \ldots, 4)$$

as a basis vector for the tangent space at point P.

Tensors in General Coordinates

Applying these new curved coordinates to our prior definitions, we find the following forms:

$$g_{ij} = \mathbf{c}_i \cdot \mathbf{c}_j = \sum_{h=1}^{N} (\partial_i X^h) \cdot (\partial_j X^h),$$

$$R_{is;jm} = \frac{1}{2}(\partial_k \partial_s g_{ij} - \partial_k \partial_i g_{sj} - \partial_j \partial_s g_{ik} + \partial_j \partial_i g_{sk}).$$

Here, we also will define the **scalar curvature**, which is the twice-contracted Riemann tensor

$$R = g^{jk} R^i_{j;ik}.$$

Additionally, we must define the **Christoffel symbols**,

$$\Gamma_{ij}^m = \frac{1}{2}g^{mk}(\partial_j g_{ki} + \partial_i g_{kj} - \partial_k g_{ij}).$$

The array of these Christoffel symbols track how the basis changes from point to point. Notably, the derivative of our component vectors become

$$\partial_j \mathbf{c}_i = \Gamma_{ij}^k \mathbf{c}_k + \mathbf{n}_{ij}.$$

Einstein's Equations

When we combine this notion of changing coordinates with our prior equations of motion, we find the following form.

$$T_{ij} = R_{ij} - \frac{1}{2}g_{ij}R \quad \text{or} \quad T_j^i = R_j^i + \frac{1}{2}\delta_{ij}R,$$

where R_{ij} is the contracted Riemann tensor, R is the total curvature, and g_{ij} is the metric tensor. These are the famous **Einstein Equations**, and reveal a surprising fact about reality. In this framework, the fundamental quantities of matter, electricity, and magnetism can all be seen as components of the structure of our curved spacetime. This also implies that, neglecting electromagnetism,

$$\partial_j T_i^j - \Gamma_{ij}^k T_k^j = 0 \quad \text{when} \quad T_j^i = \rho u^i u_j,$$

which tells us that matter subject to a gravitational field and light particles will follow geodesics. A **geodesic** can be thought of as the shortest line between two points, or in the case of our manifold it can also be thought of as the *straightest* line between two points on the manifold.

Equations of Planetary Motion

Now, consider a simplified physical system in which a single planet orbits a sun. Then, neglecting the mass of the planet itself, we find the equations

$$R_i^j - \frac{1}{2}\delta_{ij}R = 0 \quad \text{which contracts to} \quad R_j^i = 0.$$

Now, applying these general relativity formalisms to the ellipsoidal orbit of a planet, we find

$$-ds^2 = \xi(r)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) - \eta(r)dt^2.$$

This equation tells us that, in a model only containing the sun, the curved spacetime becomes a sphere with total curvature $\frac{1}{r^2}$. We can also now define our g_{ij}, with $g_{11} = \xi(r)$, $g_{22} = r^2$, $g_{33} = r^2\sin^2(\theta)$, $g_{44} = -\eta(r)$, and all others zero. We can use this to find components of our Riemann tensor,

$$R_1^1 = \frac{\xi'}{\xi^2 r} + \frac{\xi'\eta'}{4\xi^2\eta} + \frac{\eta'^2}{4\xi\eta^2} - \frac{\eta''}{2\xi\eta}$$

$$R_2^2 = R_3^3 = \frac{\xi'}{2\xi^2 r} + \frac{1}{r^2}\left(1 - \frac{1}{\xi}\right) - \frac{\eta'}{2\xi\eta r}$$

$$R_4^4 = -\frac{\xi'\eta'}{4\xi^2\eta} - \frac{\eta'^2}{4\xi\eta^2} + \frac{\eta''}{2\xi\eta} + \frac{\eta'}{\xi\eta r}$$

with all other terms vanishing. Each of these equatons must be zero, as postulated above. This results in the following conditons,

$$\xi\eta = \text{constant} \quad \text{and} \quad \eta = 1 - \frac{\gamma}{r},\ \gamma = \text{constant}.$$

Combining these with our original orbital equation, we find the equation for the motion of a planet around a star,

$$-ds^2 = \frac{1}{\eta}dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2 - \eta dt^2.$$

This equation must also be a geodesic, which means

$$-1 = \frac{\dot{r}^2}{\eta} + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2 - \eta\dot{t}^2.$$

Solving this equation and simplifying by setting $r = \frac{1}{u}$, we find

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \lambda + \frac{\gamma}{h^2}u + \gamma u^3,$$

where $\lambda = \dot{t}\eta$ and $h = \dot{\phi}r^2$ are both constants.

Conclusion

To accurately model the motion of planets and other very massive interstellar objects, the language of differential geometry and topology is required to properly express general relativity. In some sense, our classical Newtonian mechanics can be seen as a first-order approximation of general relativity, accurate for small objects relatively close to the surface of the earth. However, when we move to the scale of the cosmos, a more general theory is required.

References

[1] Rainich, G. Y. (2014). Mathematics of Relativity. Dover Publ.

[2] Penrose, R. (1972). Techniques of Differential Topology in Relativity. SIAM.

[3] Nicolaescu, L. I. (2021). Lectures on the geometry of manifolds. World Scientific Publishing Co.