Results about commutative doubly-idempotent semirings

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What is a semiring

A **semiring** $(A,+,0,\cdot,1)$ is defined by the following equations for all $x,y,z\in A$:

$$(x+y)+z=x+(y+z) \qquad (xy)z=x(yz)$$
$$x+y=y+x \qquad x1=x=1x$$
$$x+0=x \qquad x0=0=0x$$
$$x(y+z)=xy+xz \qquad (x+y)z=xz+yz$$

Example. The natural numbers $(\mathbb{N},+,0,\cdot,1)$ are a semiring.

- An **idempotent semiring** is a semiring $\mathbf{A}=(A,\vee,0,\cdot,1)$ such that $x\vee x=x$.
- **A** is **doubly idempotent** if $x\vee x=x$ and $xx=x$.
- **A** is **commutative** if $xy=yx$.

What is a semilattice

A **semilattice with 0** is an algebra $(A,\vee,0)$ such that $\vee$ is associative, commutative, idempotent ($x\vee x=x$), and $x\vee 0=x$.

A semilattice is **partially ordered** by $x\le y \iff x\vee y=y$.

A **commutative doubly-idempotent semiring** (**cdi-semiring**) is of the form $(A,\vee,0,\cdot,1)$ such that:

- $(A,\vee,0)$ is a semilattice with 0 (ordered by $\le$)
- $(A,\cdot,1)$ is a semilattice with 1 (ordered by $x\sqsubseteq y \Leftrightarrow xy=x$)
- $x0=0$, and $x(y\vee z)=xy\vee xz$ holds for all $x,y,z\in A$.

Example. All **bounded distributive lattices** are cdi-semirings where xy is the meet (greatest lower bound) of x and y.

In this case $x\le y$ if and only if $x\sqsubseteq y$.

Three subclasses of cdi-semirings

Why look at **restricted classes** of cdi-semirings, and not the whole class?

- While distributive lattices are well understood, the class of cdi-semirings is a **much bigger**. There is no general structure theory.
- The class of cdi-semirings is defined by a list of identities, hence it is a variety.
- Chajda and Länger [1] proved cdi-semirings are the smallest variety containing all bounded distributive lattices and $\mathbf{S}_3$, a 3-element semiring that is not a distributive lattice.

Consider the number of algebras for each size (up to isomorphism)[3][4]

# of elements =	1	2	3	4	5	6	7	8
# of cdi-semirings	1	1	2	6	20	77	333	1589
# of distr. lattices	1	1	1	2	3	5	8	15

Outline

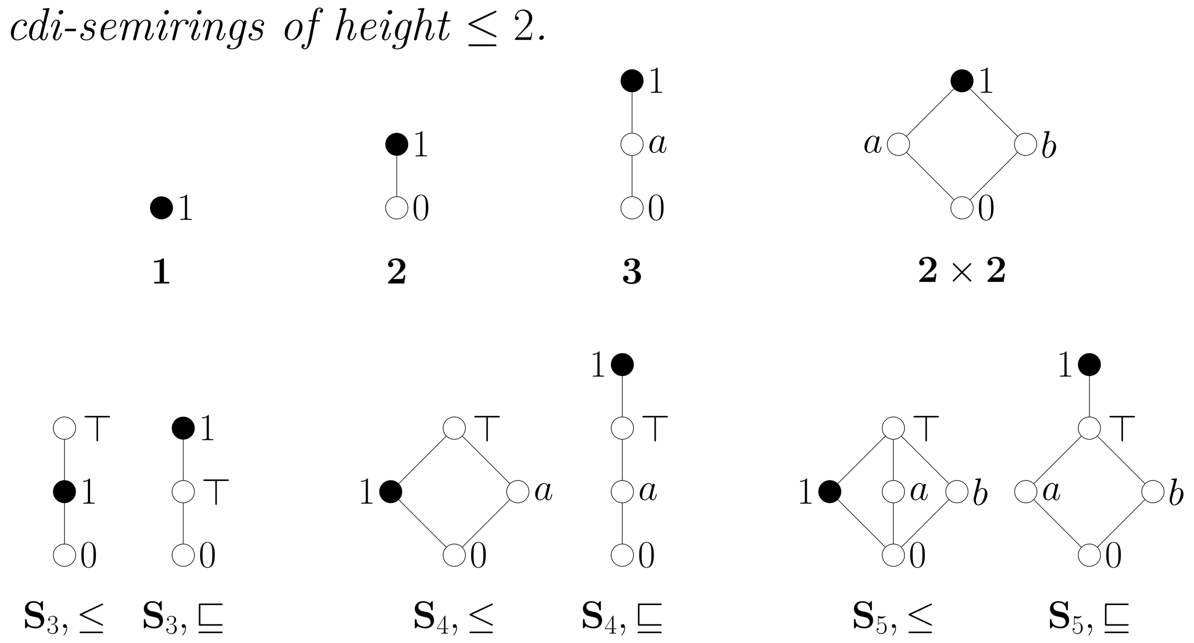
A commutative doubly-idempotent semiring (cdi-semiring) $(S,\vee,\cdot,0,1)$ is a semilattice $(S,\vee,0)$ with $x\vee 0=0$ and a semilattice $(S,\cdot,1)$ with identity 1 such that $x0=0$, and $x(y\vee z)=xy\vee xz$ holds for all $x,y,z\in S$. Bounded distributive lattices are cdi-semirings that satisfy $xy=x\wedge y$, and the variety of cdi-semirings covers the variety of distributive lattices. Chajda and Länger showed in 2018 that the variety of all cdi-semirings is generated by the 3-element cdi-semiring.
We show that there are cdi-semirings with a $\vee$-semilattice of height less than or equal to 2. We construct all cdi-semirings for which their multiplicative semilattice is a chain with $n+1$ elements, and we show that up to isomorphism the number of such algebras is the n^{th} Catalan number $C_n=\frac{1}{n+1}\binom{2n}{n}$. We also show that cdi-semirings with a complete atomic Boolean $\vee$-semilattice on the set of atoms A are determined by rooted preorder forests on the set A. From these results we obtain efficient algorithms to construct all multiplicatively linear cdi-semirings of size n and all Boolean cdi-semirings of size 2^n.

Seven cdi-semirings of height ≤ 2

The **height** of a join-semilattice is the length (# of elements -1) of the longest chain (= linear order) in it.

With the restriction on height of cdi-semirings to be less or equal to two for $(A,\vee)$, we have the following theorem.

Theorem 1. [5] *There are, up to isomorphism, seven cdi-semirings of height ≤ 2.*



Catalan semirings

A **Catalan semiring** is a multiplicatively linear cdi-semiring, i.e. $xy=x$ or $xy=y$ for all x,y.

For **A** and **B** Catalan semirings, we define the **Catalan sum** $\mathbf{C}=\mathbf{A}\copyright\mathbf{B}$ to be the structure over the disjoint union of **A** and **B**.

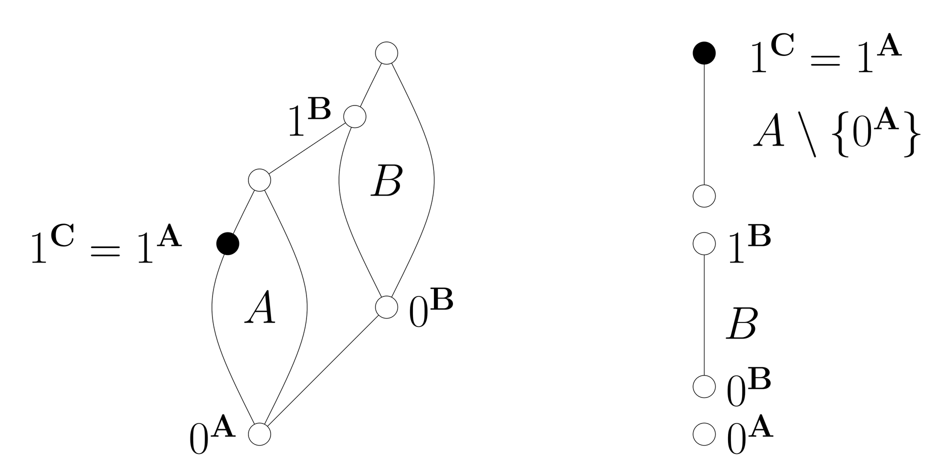


Figure 1:The Catalan sum $\mathbf{C}=\mathbf{A}\copyright\mathbf{B}$

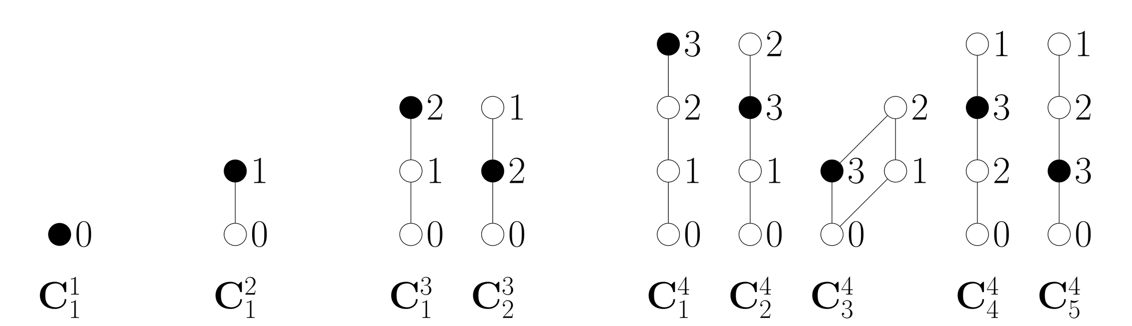
The 0, 1 of **C** are $0^{\mathbf{A}},1^{\mathbf{A}}$.

The number of Catalan semirings

Now we replace the restriction on height, with a restriction that $(A,\cdot)$ must be multiplicatively linear.

Theorem 2. [2] *The number of Catalan semirings with $n+1$ elements, up to isomorphism, is the n^{th} Catalan number*

$$C_n=\frac{1}{n+1}\binom{2n}{n}=1,1,2,5,14,42,\ldots$$



Boolean cdi-semirings

Finally, lets look at the case when we restrict $(A,\vee)$ to be a finite Boolean $\vee$-semilattice. Define R on the atoms by $R(x,y,z)\iff x\le yz$.

Then $xx=x\iff (R(x,y,z)\Rightarrow x=y$ or $x=z)$.

Define P, Q from R by

$$P(x,y)\Leftrightarrow R(x,y,x) \qquad Q(x,y)\Leftrightarrow R(x,x,y).$$

Theorem 3. *An idempotent ternary relation $R\subseteq A^3$ satisfies $(R(u,x,y)\,\&\,R(w,u,z)\Rightarrow \exists v(R(v,y,z)\,\&\,R(w,x,v)))$ if and only if $(xy)z\le x(yz)$ if and only if the corresponding reflexive relations P,Q satisfy*

(P_1) $P(x,y)\ \&\ P(y,z)\Rightarrow P(x,z)$ *P-transitivity*

(P_2) $Q(x,y)\ \&\ Q(x,z)\Rightarrow Q(y,z)$ *or* $P(z,y)$ *PQ-Euclidianess*

(P_3) $P(x,y)\ \&\ Q(y,z)\ \&\ x\ne y\Rightarrow P(x,z)$

Hence $\cdot$ is associative if and only if $(\mathsf{P}_1)-(\mathsf{P}_3)$ hold, also with P, Q interchanged.

Note that the operation $\cdot$ is commutative if $P=Q$.

Boolean cdi-semirings from preorder forests

- A **preorder** is a reflexive transitive binary relation,
- A **preorder forest** is a preorder such that
$$P(x,y)\ \&\ P(x,z)\Rightarrow P(y,z) \text{ or } P(z,y)$$
i.e., all the elements above a given element are linearly ordered.
- A preorder forest has **singleton roots** if every component has a unique top element.

Theorem 4. *Finite Boolean cdi-semirings are definitionally equivalent to finite preorder forests with singleton roots.*

- Hence all finite Boolean cdi-semirings can be constructed by enumerating preorder forests with singleton roots.

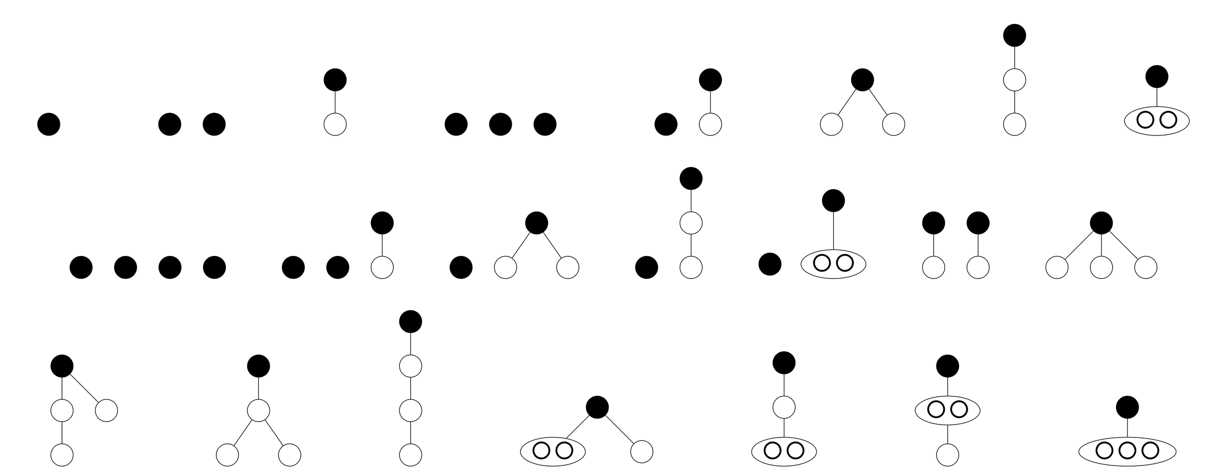


Figure 2:Preorder forests with singleton roots

Conclusion

In the theory of rings and other algebras, multiplicatively idempotent elements often play a central role in controlling some structural aspects of the algebra. The structure of idempotent semirings in general is quite challenging, but with suitable restrictions some nice characterizations can be found. Here we considered commutative doubly idempotent semirings of height ≤ 2, or with a multiplicative linear order or with a Boolean join-semilattice. In each case it was possible to give detailed descriptions of the finite members that allow them to be enumerated easily up to isomorphism. It is likely that some of the techniques explored here can be applied to larger classes of idempotent semirings by, for example, weakening the assumption of commutativity or allowing distributive join-semilattices.

References

[1] I. Chajda and H. Länger, The variety of commutative additively and multiplicatively idempotent semirings. Semigroup Forum 96 (2018), no. 2, 409–415.

[2] P. Jipsen, J. Gil-Férez, and G. Metcalfe, Structures theorems for idempotent residuated lattices, preprint.

[3] W. McCune, Prover9 and Mace4, http://www.cs.unm.edu/~mccune/Prover9, 2005–2010.

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[5] D. Stanovský, Commutative idempotent residuated lattices. Czechoslovak Math. J. 57(132) (2007), no. 1, 191–200.