

Preserving Vector Norms Using Subgaussian Random Matrices

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Abstract

We study classes of random matrices that approximately preserve Euclidean norms of high-dimensional vectors. In particular, we focus on matrices whose entries are sub-gaussian random variables. We perform numerical experiments with Gaussian, Bernoulli, and uniform random matrices to compare how well they preserve norms.

Introduction

High dimensional datasets often require dimensionality reduction for efficient computation. Random projections aim to preserve geometry of the data while reducing dimension. We study random matrices satisfying:

$$\|Ax\|_2 \approx \|x\|_2?$$

Definitions

Near Isometry. A linear map $A, A \in \mathbb{R}^{m \times n}$, is a near isometry on a set T , if there exists a small constant $\delta \geq 0$ such that for all $x \in T$,

$$\|x\|_2 - \delta \leq \|Ax\|_2 \leq \|x\|_2 + \delta$$

where δ measures the maximum distortion of the norm in the set T .

Isotropy. A random vector $X \in \mathbb{R}^n$ is called isotropic if

$$\mathbb{E}[XX^T] = I_n$$

Subgaussian Random Variable. A random variable X is called subgaussian if there exists a constant $K \geq 0$ such that for all $t \geq 0$:

$$P(|X| \geq t) \leq 2e^{-\frac{t^2}{K^2}}$$

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Theorem

Let A be an $m \times n$ matrix whose columns A_i are independent, mean zero, subgaussian random vectors in $\mathbb{R}^n$ satisfying $\|A_i\|_2 = 1$ almost surely and $\|A_i\|_{\psi_2} \leq K$. Then, for any subset $T \subset \mathbb{R}^n$, we have we have for some absolute constant $C > 0$,

$$\mathbb{E} \sup_{x \in T} \left| \|Ax\|_2 - \|x\|_2 \right| \leq CK\gamma(T)$$

Explicit example for Gaussian

Let a Gaussian matrix

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}, \quad z_{ik} \sim \mathcal{N}(1, 0).$$

Then we observe that

$$A^T \cdot A = \begin{pmatrix} z_{11}^2 + z_{12}^2 & z_{11}z_{21} + z_{12}z_{22} \\ z_{21}z_{11} + z_{22}z_{12} & z_{21}^2 + z_{22}^2 \end{pmatrix}.$$

Using properties of Gaussian Random variable,

$$\mathbb{E}[A^T \cdot A] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Then for random matrices, this becomes:

$$\begin{aligned} \mathbb{E}\|Ax\|_2^2 &= \mathbb{E}\langle Ax, Ax \rangle \\ &= \langle x, \mathbb{E}A^T A x \rangle = \langle x, Ix \rangle = \|x\|_2^2 \end{aligned}$$

Example 1: Gaussian matrix

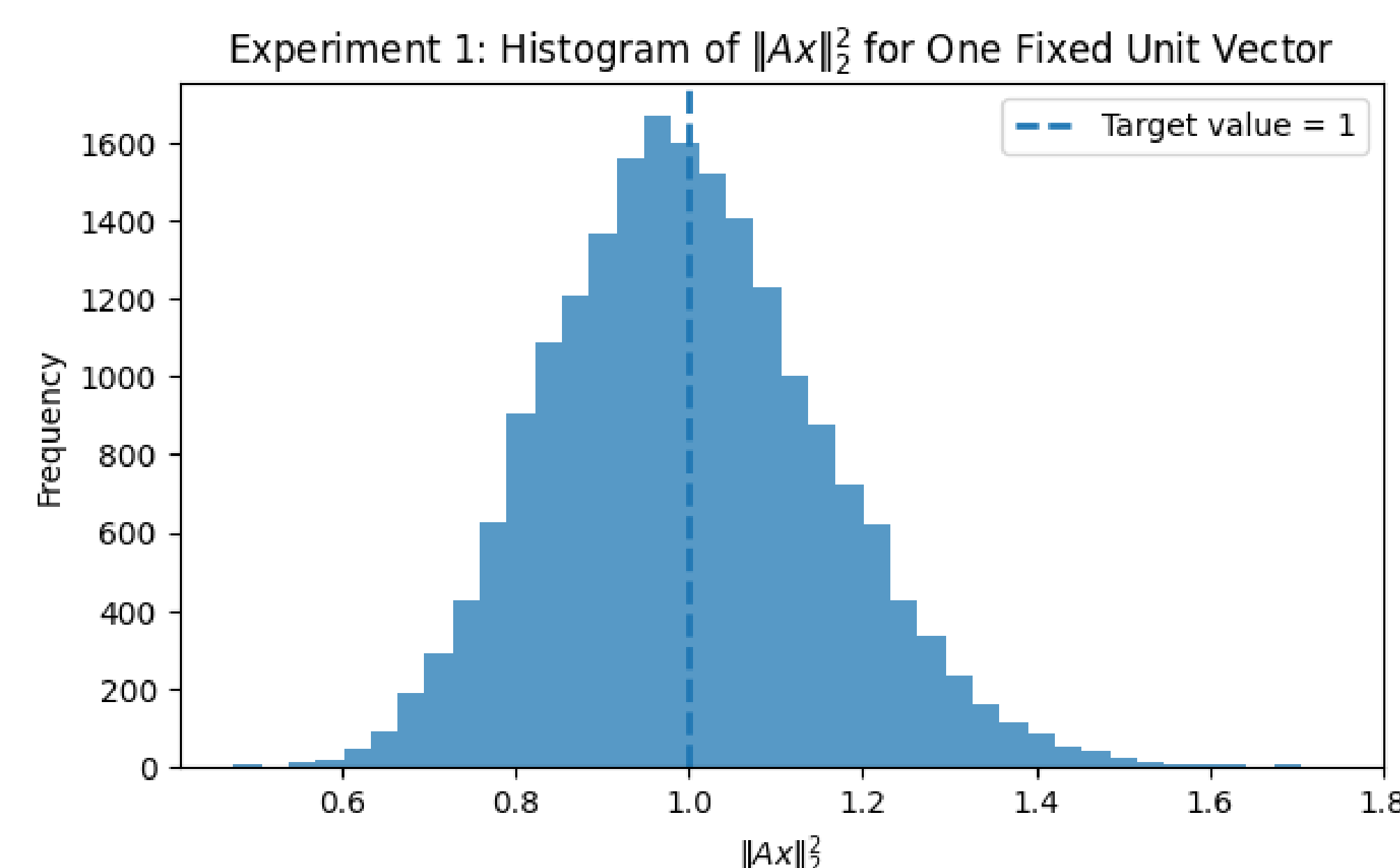
We study the random matrix

$$A = \frac{1}{\sqrt{m}}Z,$$

where Z is an $m \times n$ matrix whose entries are independent standard Gaussian random variables:

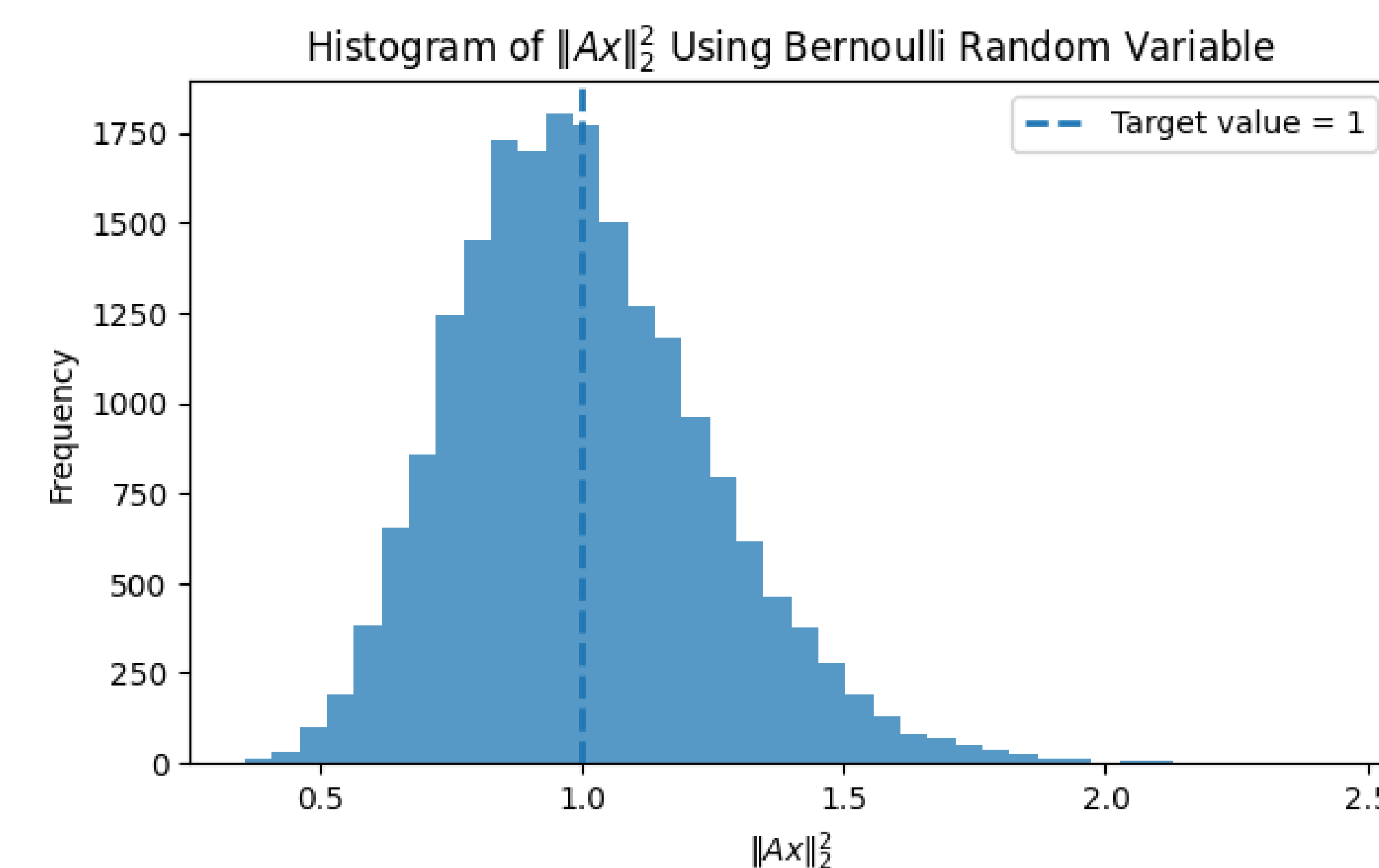
$$z_{ij} \sim N(0, 1).$$

This is the basic toy example behind random projections, near-isometries, and the Johnson–Lindenstrauss idea.



Example 2: Bernoulli Matrix

Let $A \in \mathbb{R}^{m \times n}$ be a matrix whose elements are independent Bernoulli random variable X where $\mathbb{P}(X = 1) = \frac{1}{m}$. A is constructed such that $A^T A \approx I$. However, the lack negative values mean that the Bernoulli matrix has a tendency to overstate the norm.



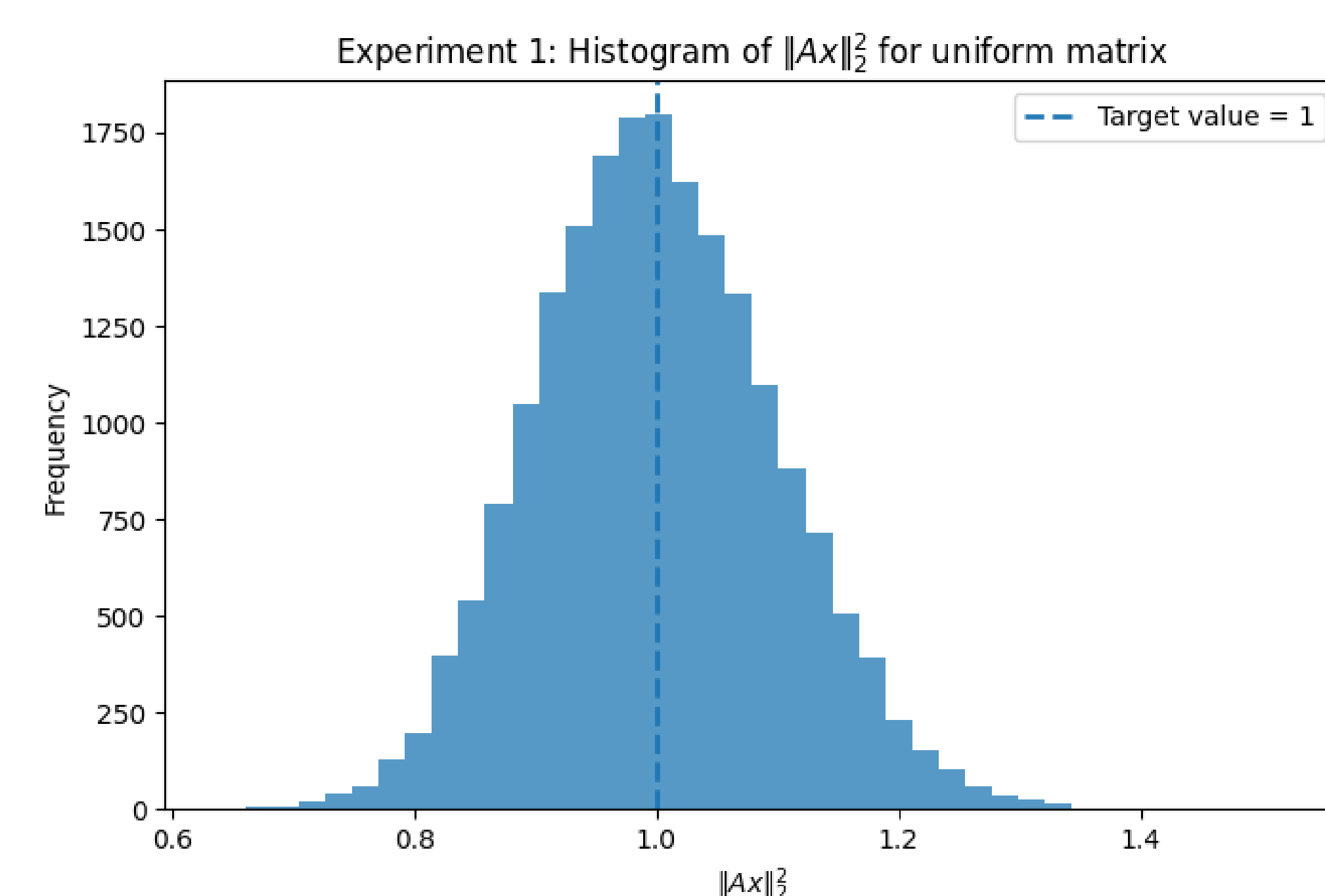
Example 3: Uniform matrix

We consider a matrix A such that:

$$A = \frac{1}{\sqrt{m}}Z$$

where Z is an $m \times n$ matrix with independent entries:

$$z_{ij} \sim \text{Uniform}(-\sqrt{3}, \sqrt{3})$$



Example 4: Uniform matrix (Fails)

We study the random matrix:

$$A = \frac{1}{\sqrt{m}}Z$$

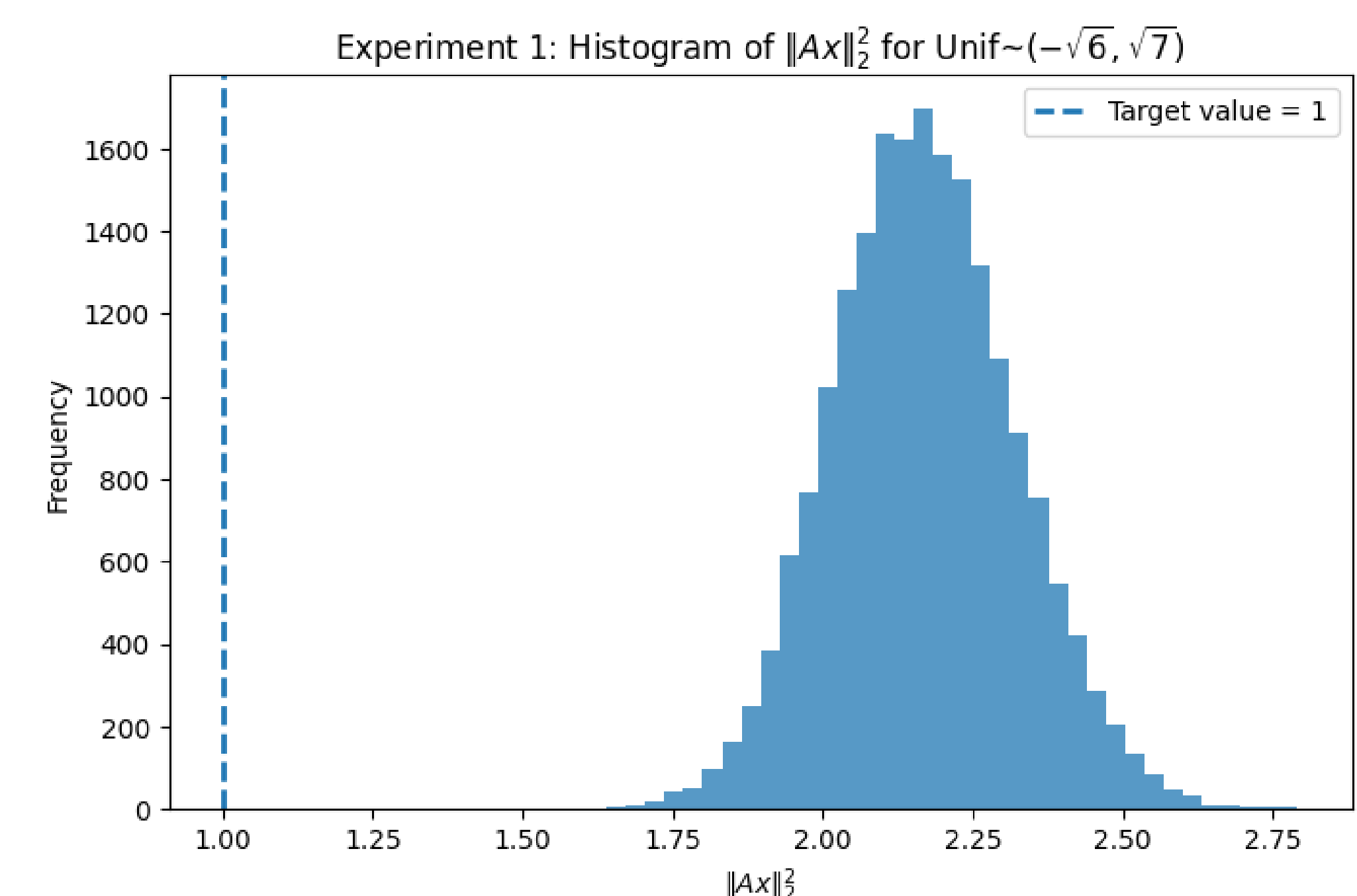
where Z is an $m \times n$ matrix with independent entries:

$$z_{ij} \sim \text{Uniform}(-\sqrt{6}, \sqrt{7})$$

For each entry z_{ij} :

$$\mathbb{E}[z_{ij}] = 0.0981, \quad \text{Var}(z_{ij}) = 2.1634$$

After scaling Z by $\frac{1}{\sqrt{m}}$, we observe the following:



Conclusion

Our examples illustrate that Gaussian, sub-Gaussian, and appropriately normalized uniform matrices preserve Euclidean norms. However, not all random matrices are norm preserving as illustrated by Example 4. Furthermore, Example 2 shows that certain subgaussian matrices will have less optimal performances. Future work is needed to find how to improve the accuracy of preserving norms and studying the distortion of caused by the random matrix.

References

R. Vershynin. High-Dimensional Probability: An Introduction with Applications in Data Science. 2nd ed. Cambridge University Press; 2026.

Figure 1: The norm of $\|Ax\|_2^2$ when taking unit vector $\|x\| = 1$.