

Rademacher Random Matrices as Near-Isometries on the Unit Sphere

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Introduction

In this poster, we explore for which classes of random matrices A do we have

$$\|Ax\|_2 \approx \|x\|_2?$$

That is: which random matrices behave like *near-isometries*?

Motivation

High dimensional datasets often require dimension reduction for efficient computation. Random projections aim to preserve geometry of the data while reducing dimension.

Definitions

- A **random vector** in $\mathbb{R}^n$ is a vector $X = (X_1, \dots, X_n)$, where each component X_i is a real-valued random variable. A **random matrix** has random vectors as its rows/columns.
- The **subgaussian norm** of a random variable Z is defined by

$$\|Z\|_{\psi_2} := \inf \left\{ t > 0 : \mathbb{E} \exp \left(\frac{Z^2}{t^2} \right) \leq 2 \right\}.$$

- A random variable Z is **subgaussian** if

$$\|Z\|_{\psi_2} < \infty$$

- A random vector $X := (X_1, \dots, X_n)$ is a **subgaussian vector** if for every unit vector $u \in \mathbb{R}^n$, the real-valued random variable $\langle X, u \rangle$ is subgaussian.

- A random vector $X \in \mathbb{R}^n$ is **isotropic** if

$$\mathbb{E}[XX^T] = I_n,$$

where I_n is the identity matrix.

- The **isometric constant** δ of an $m \times n$ random matrix A acting on a set $T \subseteq \mathbb{R}^n$ is defined by

$$\delta := \sup_{x \in T} |\|Ax\|_2 - \|x\|_2|$$

Some Subgaussian Random Variables

A **Rademacher random variable** X satisfies:

$$\mathbb{P}(X = 1) = \frac{1}{2}, \quad \mathbb{P}(X = -1) = \frac{1}{2}.$$

Uniform random variables are also subgaussian: let $X \sim \text{Unif}[-1, 1]$. Then $|X| \leq 1$ almost surely, and by Hoeffding's lemma,

$$\mathbb{E}[e^{tX}] \leq e^{t^2/2} \Rightarrow \|X\|_{\psi_2} < \infty.$$

Rademacher Variables are Subgaussian

Let X be a Rademacher random variable. Since $X^2 = 1$ almost surely, we have $\mathbb{E} \exp(X^2/t^2) = \mathbb{E} \exp(1/t^2) = \exp(1/t^2)$, so we need $\exp(1/t^2) \leq 2$. Taking logarithms, $\frac{1}{t^2} \leq \log 2$, which is equivalent to $t \geq \frac{1}{\sqrt{\log 2}}$. Hence,

$$\|X\|_{\psi_2} = \frac{1}{\sqrt{\log 2}} < \infty.$$

Binomial Random Variable

Another example of a subgaussian random variable is a *Binomial Random Variable*. This is a random variable X satisfying:

$$X = \sum_{i=1}^n B_i \quad B_i \sim \text{Ber}(p)$$

Let X be a Binomial random variable. Since $X^2 \leq n^2$ we have $\mathbb{E} \exp(X^2/t^2) \leq \exp(n^2/t^2)$ so we need $\exp(n^2/t^2) \leq 2$. Taking logarithms, $\frac{n^2}{t^2} \leq \log 2$, which is equivalent to $t \geq \frac{n}{\sqrt{\log 2}}$. Hence,

$$\|X\|_{\psi_2} \leq \frac{n}{\sqrt{\log 2}} < \infty.$$

Conclusion

- Our experiments illustrate the concentration behavior predicted by Theorem 1.2; after normalization, Rademacher matrices approximately preserve Euclidean lengths with high probability.
- These computations provide a concrete demonstration of the theoretical guarantees developed by Plan and Vershynin for subgaussian random matrices acting on subsets of $\mathbb{R}^n$.

References

R. Vershynin. *High-Dimensional Probability: An Introduction with Applications in Data Science*. 2nd ed. Cambridge University Press; 2026.

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Abstract

We study random matrices with Rademacher entries as an instance of subgaussian random matrices. Our goal is to understand numerically whether these matrices behave like near-isometries on unit vectors, by tracking the empirical distribution of the distortions $|\|Ax\|_2 - \|x\|_2|$ over randomly sampled unit vectors.

Matrices with independent, isotropic, subgaussian rows

Theorem 1.2 (Plan and Vershynin, “Random Matrices Acting on Sets: Independent Columns”).

Let A be an $m \times n$ matrix whose rows are independent, isotropic, subgaussian random vectors in $\mathbb{R}^n$ with subgaussian norms bounded by K . Then, for any subset $T \subset \mathbb{R}^n$, we have

$$\mathbb{E} \sup_{x \in T} |\|Ax\|_2 - \sqrt{m} \|x\|_2| \leq CK \sqrt{\log K} \gamma(T).$$

- Normalizing the matrix $\tilde{A} := \frac{1}{\sqrt{m}}$, from Theorem 1.2, we have

$$\mathbb{E}[\|\tilde{A}x\|_2^2] = 1$$

for any fixed unit vector $x \in S^{n-1}$. Thus $\tilde{A}$ behaves like a near-isometry.

- Since

$$\|\tilde{A}x\|_2^2 = \frac{1}{m} \sum_{i=1}^m \langle A_i, x \rangle^2$$

is an average of m independent random variables with mean 1, it *concentrates near 1* as the m grows large.

Numerical Experiments

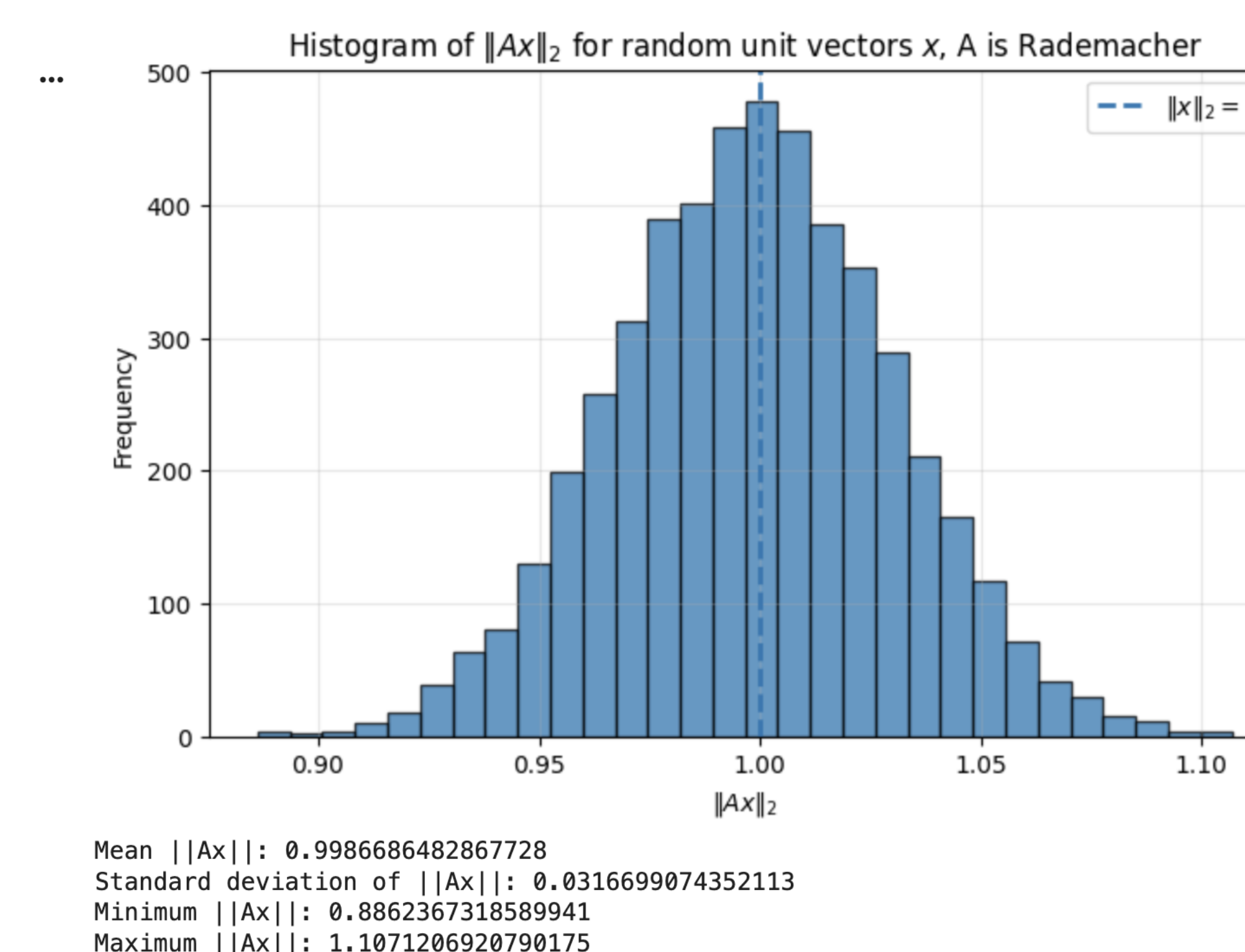


Figure 1: The distribution of $\|Ax\|_2$.

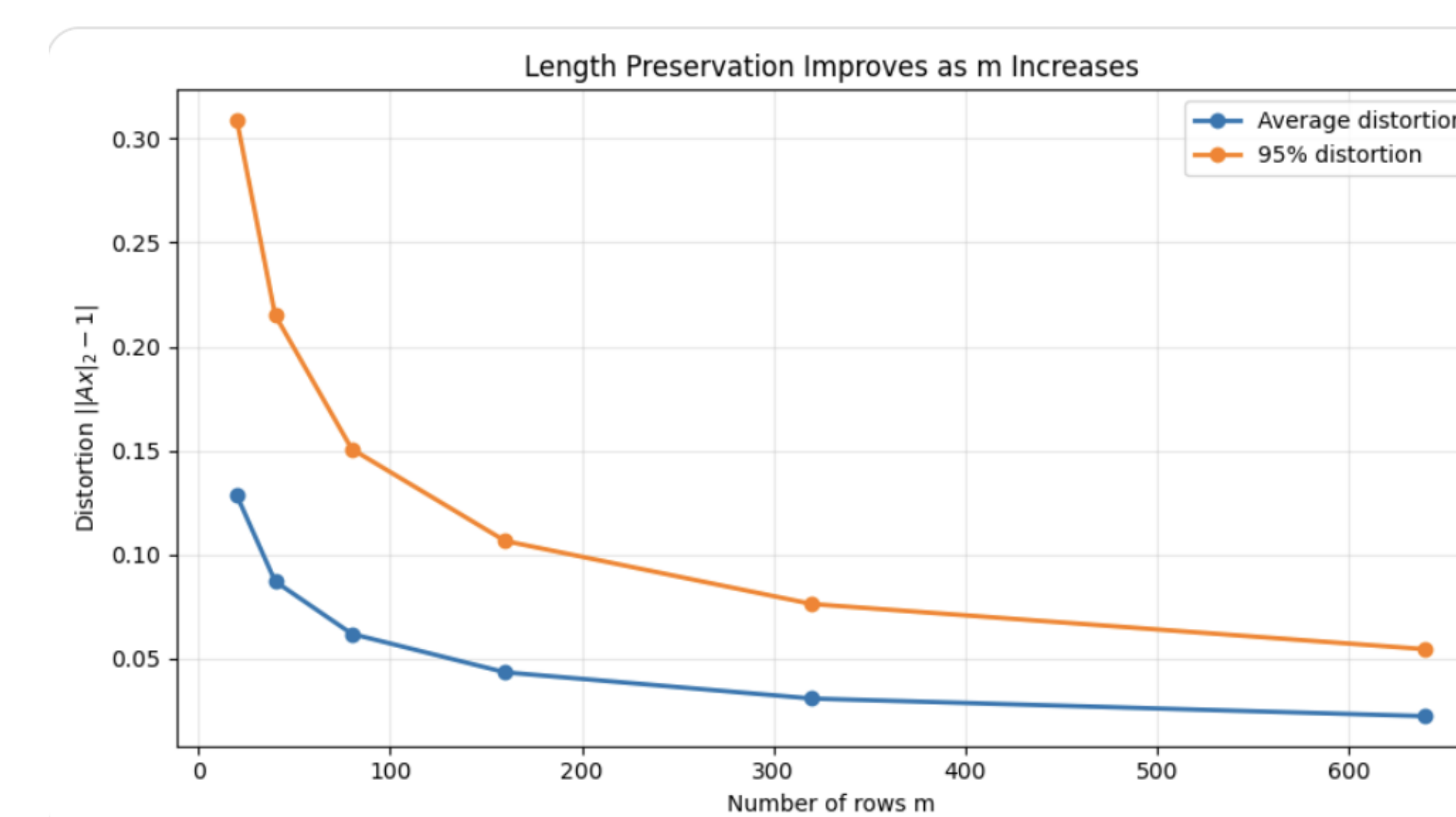


Figure 2: Distortion decreases as m increases