The structure of distributive idempotent lattice-ordered magmas

CHAPMAN UNIVERSITY
SCHMID COLLEGE OF SCIENCE AND TECHNOLOGY

Natanael Alpay and Peter Jipsen

Chapman University

What is a distributive lattice

A **lattice** is an algebra $(A, \wedge, \vee)$ defined by the following equations for all $x, y, z \in A$

$$(x \vee y) \vee z = x \vee (y \vee z) \qquad (x \wedge y) \wedge z = x \wedge (y \wedge z)$$
$$x \vee y = y \vee x \qquad x \wedge y = y \wedge x$$
$$x \vee (x \wedge y) = x \qquad x \wedge (x \vee y) = x$$

It is **distributive** if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

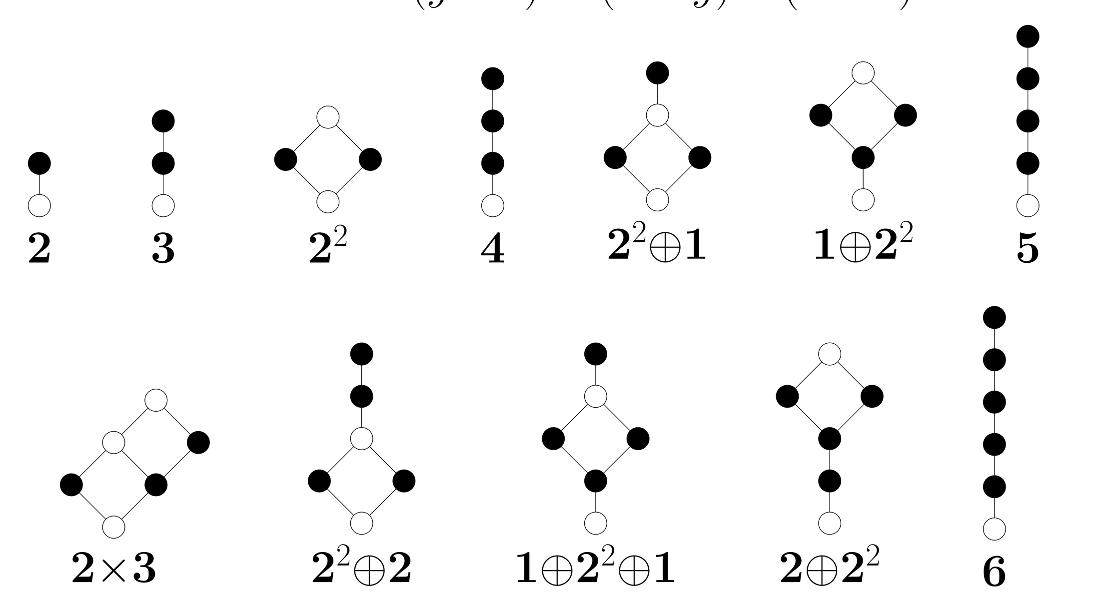


Figure 1: Distributive lattices of size 6 or less with join-irreducibles in black.

A lattice is **bounded** if it has a bottom and a top element.

In a lattice, x is a **complement** of y if $x \wedge y$ is the bottom element and $x \vee y$ is the top element.

A bounded distributive lattice is **Boolean** if every element has a complement.

A lattice is **complete** if $\wedge S$ and $\vee S$ exist for all $S \subseteq A$.

Join-irreducibles and partial orders

x is **completely join-irreducible** if $x = \vee S \implies x \in S$.

Let $J(A)$ denote the set of completely join-irreducibles of A.

If A is a Boolean lattice, then $J(A) = At(A)$ which is the set of all elements immediately above the bottom element.

A lattice is **perfect** if every element is a join of completely join-irreducibles and a meet of completely meet-irreducibles.

$(W, \leq)$ is a **partially-ordered set** if for all $x, y, z \in W$: $x \leq x$ (reflexivity), $x \leq y$ and $y \leq x$ implies $y = x$ (antisymetric), $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitivity).

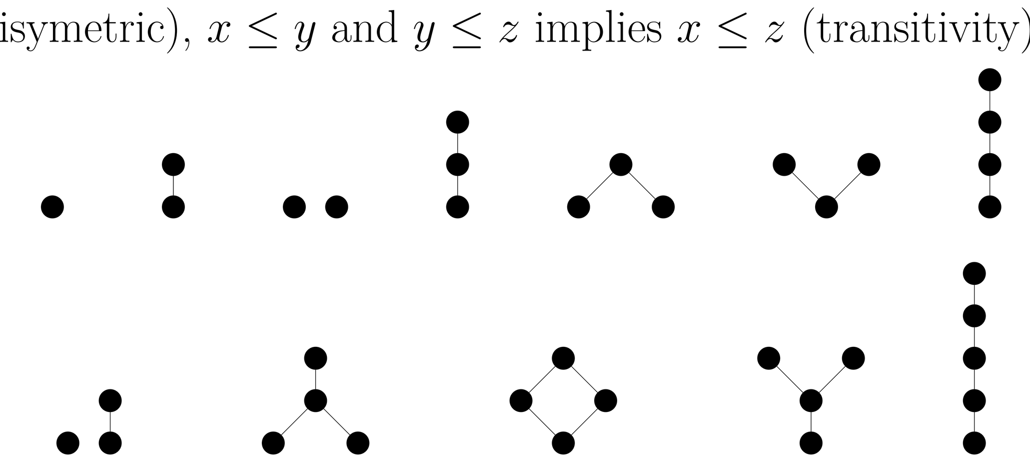


Figure 2:Partially ordered sets of join-irreducibles.

A **downset** is a subset X such that $y \leq x \in X$ implies $y \in X$.

Let $D(W, \leq)$ be the set of all downsets.

The **lattice of downsets** is $(D(W, \leq), \cap, \cup)$.

Theorem 1. *A distributive lattice A is complete and perfect if and only if it is isomorphic to the lattice of downsets of a partial order.*

Outline

A **lattice-ordered magma** (**ℓ-magma** for short) $(A, \wedge, \vee, 0, \cdot)$ is a lattice with 0 and a binary operation $\cdot$ such that $x0 = 0 = 0x$, $x(y \vee z) = xy \vee xz$, $(x \vee y)z = xz \vee yz$, and $x \vee 0 = x$ hold for all $x, y, z \in A$.
A **distributive idempotent ℓ-magma** (or diℓ-magma) is an ℓ-magma A that satisfies $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $xx = x$.
Let $J(A)$ be the set of completely join-irreducible elements of A, and define the property of **weakly conservative** as

$$xy = x \wedge y \quad \text{or} \quad xy = x \quad \text{or} \quad xy = y \quad \text{or} \quad xy = x \vee y$$

for all $x, y \in J(A)$.
We show that every complete perfect weakly conservative diℓ-magma A is determined by two binary relations on the partially-ordered set $J(A)$. If two binary relation coincide and satisfy preorder forest axioms then we obtain dℓ-semilattices.
From these results we obtain efficient algorithms to construct all weakly conservative diℓ-magmas and dℓ-semilattices of size n.

What is a lattice-ordered magma

A **lattice-ordered magma** (**ℓ-magma** for short) $(A, \wedge, \vee, \cdot, 0)$ is a lattice with a binary operation $\cdot$ and 0 such that for all $x, y, z \in A$

$$x0 = 0 \qquad x(y \vee z) = xy \vee xz$$
$$0x = 0 \qquad (x \vee y)z = xz \vee yz$$
$$x \vee 0 = x$$

An ℓ-magma is **associative** if for all x, y, z, $(xy)z = x(yz)$, and **commutative** if $xy = yx$.

An ℓ-magma is **idempotent** if $xx = x$.

This is equivalent to $x \wedge y \leq xy \leq x \vee y$.

An **ℓ-semilattice** is an ℓ-magma that is associative, commutative and idempotent.

Birkhoff frames

Let A be a distributive complete prefect ℓ-magma. Then define $(J(A), \leq, R)$ to be the **Birkhoff frame** of A where the ternary relation R is given by $R(x, y, z) \iff x \leq yz$.

From the definition of ℓ-magama, $\cdot$ is order preserving, R is **down-up-up-closed**, which means that for x, x', y, y', z, z'

$$R(x, y, z) \;\&\; x' \leq x \;\&\; y \leq y' \;\&\; z \leq z' \implies R(x', y', z')$$

More generally, define a Birkhoff frame $(W, \leq, R)$, where $(W, \leq)$ is a poset and $R \subseteq W^3$ is down-up-up-closed.

For a Birkhoff frame $\mathbf{W}$ define the **downset algebra** $D(\mathbf{W}) = (D(W, \leq), \cap, \cup, \cdot, \emptyset)$, where for $Y, Z \in D(W, \leq)$

$$Y \cdot Z = \{x \in W \mid R(x, y, z) \text{ for some } y \in Y \text{ and } z \in Z\}.$$

Note that $Y \cdot Z$ is a downset by the down-up-up property of R.

Theorem 2. *Let $\mathbf{W}$ be a Birkhoff frame. Then*

- *$D(\mathbf{W})$ is a distributive complete perfect ℓ-magma.*
- *$D(\mathbf{W})$ is associative if and only if* $\exists u (R(u, x, y) \,\&\, R(w, u, z)) \iff \exists v (R(v, y, z) \,\&\, R(w, x, v))$.
- *$D(\mathbf{W})$ is commutative if and only if* $R(x, y, z) \iff R(x, z, y)$.
- *$D(\mathbf{W})$ is idempotent if and only if for all $x, y, z \in W$, $R(x, x, x)$, and $(R(x, y, z) \implies x \leq y$ or $x \leq z)$.*

Weakly conservative

A binary operation is **conservative** if it satisfies for all $x, y \in A$

$$xy = x \text{ or } xy = y.$$

A perfect ℓ-magma is called **weakly conservative** if it satisfies the universal formula for all $x, y \in J(A)$

$$xy = x \wedge y \text{ or } xy = x \text{ or } xy = y \text{ or } xy = x \vee y.$$

A Birkhoff frame $(W, \leq, R)$ is **weakly conservative**, if for all $x, y, z \in W$, $x \leq y \implies R(x, x, y) \;\&\; R(x, y, x)$ and $R(x, y, z) \iff$
$x \leq y \;\&\; x \leq z$ or $x \leq y \;\&\; R(y, y, z)$ or $x \leq z \;\&\; R(z, y, z)$.

Theorem 3. *A Birkhoff frame $\mathbf{W}$ is weakly conservative if and only if $D(\mathbf{W})$ is weakly conservative.*

PQ-frames

$(W, \leq, P, Q)$ is a **PQ-frame** if

1. $(W, \leq)$ is a poset.
2. $P(x, y) \;\&\; x \leq u \;\&\; x \not\leq v \;\&\; y \leq v \implies P(u, v)$
3. $Q(x, y) \;\&\; x \leq u \;\&\; x \not\leq v \;\&\; y \leq v \implies Q(u, v)$
4. $x \leq y \implies P(x, y) \;\&\; Q(x, y)$

Theorem 4. *Let $(W, \leq, P, Q)$ be a PQ-frame, and define $R(x, y, z) \iff$
$x \leq y \;\&\; x \leq z$ or $x \leq y \;\&\; Q(y, z)$ or $x \leq z \;\&\; P(z, y)$.
Then $(W, \leq, R)$ is a weakly conservative Birkhoff frame.*

Theorem 5. *Let $(W, \leq, R)$ be a weakly conservative Birkhoff frame and define*

$$P(x, y) \iff R(x, y, x) \quad \text{and} \quad Q(x, y) \iff R(x, x, y).$$

Then $(W, \leq, P, Q)$ is a PQ-frame.

P-frames

A **P-frame** is a PQ-frame where $P = Q$.

P is **transitive** if $P(x, y) \;\&\; P(y, z) \implies P(x, z)$.

A P-frame is a **preorder forest** if it is transitive and $P(x, y) \;\&\; P(x, z) \implies P(y, z)$ or $P(z, y)$.

Theorem. *Let $(W, \leq, P)$ be a P-frame and define*

$$R(x, y, z) \iff x \leq y, z \text{ or } x \leq y \& P(y, z) \text{ or } x \leq z \& P(z, y).$$

If P is a preorder forest, then

$$\exists u (R(u, x, y) \,\&\, R(w, u, z)) \iff \exists v (R(v, y, z) \,\&\, R(w, x, v)).$$

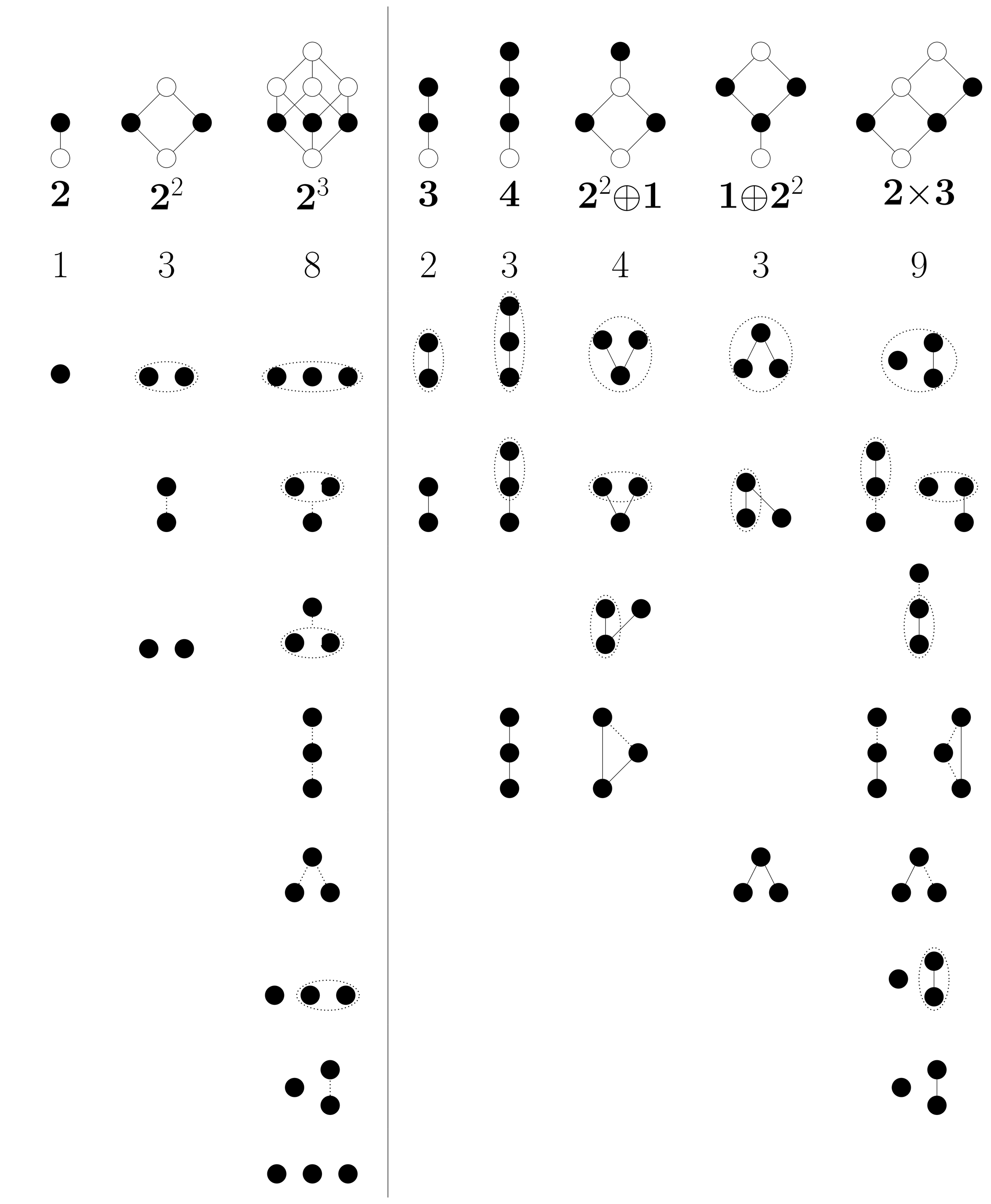
Conclusion

The point of the previous result is that it allows the construction of complete distributive perfect ℓ-semilattices from preorder forests that contain a partial order.

# of elements $n =$	1	2	3	4	5	6
# of preorder forests	1	3	8	24	71	229
# of preorder forest P-frames	1	5	27	182		

If a complete distributive perfect ℓ-semilattice has an identity element then it corresponds to a commutative distributive idempotent residuated lattice.

All 33 preorder forest P-frames $(W, \leq, P)$ with up to 3 join-irreducibles. Solid lines are the poset $(W, \leq)$, and dotted lines indicate the additional edges of the preorder P. On the left are the antichain preorder forests from [1].



References

[1] N. Alpay, P. Jipsen: *Commutative doubly-idempotent semirings determined by chains and by preorder forests*, in proceedings of the 18th International Conference on Relational and Algebraic Methods in Computer Science (RAMiCS), LNCS Vol 12062, Springer (2020), 1–14.

[2] P. Jipsen, Mathematical Structures

[3] P. Jipsen, J. Gil-Férez, and G. Metcalfe, Structures theorems for idempotent residuated lattices, preprint.

[4] W. McCune, Prover9 and Mace4, http://www.cs.unm.edu/~mccune/Prover9, 2005–2010.